Lec 7

Matrix Exponentials

1. Definition of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

The above series converges for any real number x.

2. Definition of e^A:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

The above series converges for any square matrix A.

Example 7.1:

$$A = \begin{pmatrix} -3 & & 0 \\ & -4 & \\ 0 & & -5 \end{pmatrix}$$

$$e^A = \begin{pmatrix} e^{-3} & & 0 \\ & e^{-4} & \\ 0 & & e^{-5} \end{pmatrix}$$

Example 7.2:

Nilpotent matrix.

If $A^{n-1} \neq 0$ & $A^n = 0$ then

$$e^A = I + A + \frac{A^2}{2!} + \cdots + \frac{A^{n-1}}{(n-1)!}$$

"Finite sum"

Example 7.3:

Let
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then
$$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \& \quad A^3 = 0$$

$$e^A = I + A + \frac{A^2}{2!}$$

$$= \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 7.4:

If A & B are two $n\times n$ matrices such that $AB = BA$ then

$$e^{A+B} = e^{A} e^{B}. \quad (*)$$

Caution: In general $(*)$ is not true.

If $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Show that $AB = BA = \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix}$

Hence

$$e^{\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}} = \begin{pmatrix} e^{\lambda} & 0 & 0 \\ 0 & e^{\lambda} & 0 \\ 0 & 0 & e^{\lambda} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda} & e^{\lambda} & \frac{1}{2} e^{\lambda} \\ 0 & e^{\lambda} & e^{\lambda} \\ 0 & 0 & e^{\lambda} \end{pmatrix}$$

3. Let A and B be two similar $n \times n$ matrices i.e. $\exists T$:

$$B = T^{-1} A T$$

Fact:

$$\boxed{e^{B} = T^{-1} e^{A} T}$$

Proof:

$$e^{B} = e^{T^{-1}AT}$$

$$= I + T^{-1}AT + \frac{(T^{-1}AT)^2}{2!} + \frac{(T^{-1}AT)^3}{3!} + \cdots$$

However

$$(T^{-1}AT)^2 = (T^{-1}AT)(T^{-1}AT)$$

$$= T^{-1}A(TT^{-1})AT$$

$$= T^{-1} A I A T$$

$$= T^{-1} A . A T$$

$$= T^{-1} A^2 T$$

In general $(T^{-1}AT)^n = T^{-1} A^n T$

$$e^B =$$

$$I + T^{-1}AT + \frac{T^{-1}A^2T}{2!} + \frac{T^{-1}A^3T}{3!} + \cdots\cdots$$

$$= T^{-1}\left[I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\cdots\right]T$$

$$= T^{-1}e^A T$$

$\blacksquare$

Remark: For reasons that will be clear later, (when we are solving differential equations), we are interested in calculating e^{At}, instead of e^A. "t stands for time"

$$\boxed{e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\cdots}$$

Example 7.5:

Let B be the following matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & -47 & -12 \end{pmatrix}$$

We want to calculate e^{Bt}.

7.8

```
>> B=[0 1 0;0 0 1;-60 -47 -12]
```

B =

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -60 & -47 & -12 \end{pmatrix}$$

```
>> [v j]=eig(B)
```

v =

$$\begin{pmatrix} 0.1048 & -0.0605 & -0.0392 \\ -0.3145 & 0.2421 & 0.1960 \\ 0.9435 & -0.9684 & -0.9798 \end{pmatrix}$$

scaled eigenvectors

$$V = \begin{pmatrix} 1 & 1 & 1 \\ -3 & -4 & -5 \\ 9 & 16 & 25 \end{pmatrix}$$

Three eigenvectors

j =

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\boxed{B = V j V^{-1}}$$

Linear Algebra.

diag	**Create or extract diagonals.**
triu	**Upper triangle.**
tril	**Lower triangle.**
inv	**Matrix inverse.**
det	**Determinant.**
rank	**Rank.**
rref	**Reduced row echelon form.**
null	**Basis for null space.**
colspace	**Basis for column space.**
eig	**Eigenvalues and eigenvectors.**
svd	**Singular values and singular vectors.**
jordan	**Jordan canonical (normal) form.**
poly	**Characteristic polynomial.**
expm	**Matrix exponential.**

```
>> poly(B)

ans =

   1.0000   12.0000   47.0000   60.0000
```

$\lambda^3 + 12\lambda^2 + 47\lambda + 60$

```
>> help expm

 EXPM  Matrix exponential.
   EXPM(X) is the matrix exponential of X.  EXPM is computed using
   a scaling and squaring algorithm with a Pade approximation.

   Although it is not computed this way, if X has a full set
   of eigenvectors V with corresponding eigenvalues D, then
   [V,D] = EIG(X) and EXPM(X) = V*diag(exp(diag(D)))/V.

   EXP(X) computes the exponential of X element-by-element.

 Overloaded methods
   help sym/expm.m

>> help sym/expm.m

 EXPM  Symbolic matrix exponential.
   EXPM(A) is the matrix exponential of the symbolic matrix A.

   Examples:
     syms t
     A = [0 1; -1 0]
     expm(t*A)
```

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

```
  Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

  To get started, select "MATLAB Help" from the Help menu.
```

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```
>> B=[0 1 0;0 0 1;-60 -47 -12]

B =

     0     1     0
     0     0     1
   -60   -47   -12

>> [v j]=eig(B)

v =

    0.1048   -0.0605   -0.0392
   -0.3145    0.2421    0.1960
    0.9435   -0.9684   -0.9798


j =

   -3.0000         0         0
         0   -4.0000         0
         0         0   -5.0000

>> inv(v)*j*v

ans =

    1.5000   -1.1547    0.9347
   10.3923   -3.0000    6.4758
   -4.0120   -3.0884  -10.5000
```

Wrong formula

```
>> v*j*inv(v)

ans =

   -0.0000    1.0000    0.0000
   -0.0000   -0.0000    1.0000
  -60.0000  -47.0000  -12.0000
```

$v j v^{-1}$ is the right formula

```
>> syms t
>> H=expm(t*B)
```

A little symbolic calculation

```
H =

[          6*exp(-5*t)-15*exp(-4*t)+10*exp(-3*t),          9/2*exp(-3*t)-8*exp(-4*t)+7/2*exp(-5
*t),          1/2*exp(-3*t)-exp(-4*t)+1/2*exp(-5*t)]
[       -30*exp(-3*t)+60*exp(-4*t)-30*exp(-5*t),   -35/2*exp(-5*t)+32*exp(-4*t)-27/2*exp(-3
*t),       -3/2*exp(-3*t)+4*exp(-4*t)-5/2*exp(-5*t)]
[     -240*exp(-4*t)+150*exp(-5*t)+90*exp(-3*t), 81/2*exp(-3*t)-128*exp(-4*t)+175/2*exp(-5
*t),     25/2*exp(-5*t)+9/2*exp(-3*t)-16*exp(-4*t)]

>> J=[-3 0 0;0 -4 0;0 0 -5]

J =
```

J was retyped to make the elements an integer.

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -5 \end{pmatrix} = J$$

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```
>> H1=expm(t*J)

H1 =

[ exp(-3*t),          0,          0]
[         0, exp(-4*t),          0]
[         0,          0, exp(-5*t)]
```

$H1 = e^{Jt}$

```
>> V=[1 1 1;-3 -4 -5;9 16 25]

V =
```

$$\begin{pmatrix} 1 & 1 & 1 \\ -3 & -4 & -5 \\ 9 & 16 & 25 \end{pmatrix}$$

columns are eigenvectors of B. The eigenvectors are scaled so that they are integers.

```
>> V*J*inv(V)

ans =
```

$$\begin{pmatrix} 0.0000 & 1.0000 & 0.0000 \\ -0.0000 & -0.0000 & 1.0000 \\ -60.0000 & -47.0000 & -12.0000 \end{pmatrix}$$

$V J V^{-1} = B$

```
>> V*H1*inv(V)

ans =

[          6*exp(-5*t)-15*exp(-4*t)+10*exp(-3*t),        9/2*exp(-3*t)-8*exp(-4*t)+7/2*exp(-5 
*t),        1/2*exp(-3*t)-exp(-4*t)+1/2*exp(-5*t)]
[       -30*exp(-3*t)+60*exp(-4*t)-30*exp(-5*t),   -35/2*exp(-5*t)+32*exp(-4*t)-27/2*exp(-3 
*t),      -3/2*exp(-3*t)+4*exp(-4*t)-5/2*exp(-5*t)]
[     -240*exp(-4*t)+150*exp(-5*t)+90*exp(-3*t), 81/2*exp(-3*t)-128*exp(-4*t)+175/2*exp(-5 
*t),     25/2*exp(-5*t)+9/2*exp(-3*t)-16*exp(-4*t)]

>>
```

$$V H1 V^{-1} = V e^{Jt} V^{-1}$$

$$= e^{V J V^{-1} t}$$

$$= e^{Bt}$$

$$\begin{pmatrix} 6e^{-5t}-15e^{-4t}+10e^{-3t} & 9/2e^{-3t}-8e^{-4t}+7/2e^{-5t} & 1/2e^{-3t}-e^{-4t}+1/2e^{-5t} \\ -30e^{-3t}+60e^{-4t}-30e^{-5t} & -35/2e^{-5t}+32e^{-4t}-27/2e^{-3t} & -3/2e^{-3t}+4e^{-4t}-5/2e^{-5t} \\ -240e^{-4t}+150e^{-5t}+90e^{-3t} & 81/2e^{-3t}-128e^{-4t}+175/2e^{-5t} & 25/2e^{-5t}+9/2e^{-3t}-16e^{-4t} \end{pmatrix}$$

Example 1·6 :

(7·12)

```
Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

To get started, select "MATLAB Help" from the Help menu.

>> j=[-3 1 0;0 -4 1;0 0 -5]

j =

    -3     1     0
     0    -4     1
     0     0    -5
```

Eigenvalues are distinct and are at −3, −4, −5.

This is not in the jordan form. It would be if the eigenvalues are repeated.

```
>> [T Diag]=eig(j)

T =

    1.0000   -0.7071    0.3333
         0    0.7071   -0.6667
         0         0    0.6667
```

Linearly independent set of eigenvectors

```
Diag =

    -3     0     0
     0    -4     0
     0     0    -5
```

TjT^{-1}

```
>> syms t
>> expm(t*j)

ans =

[                        exp(-3*t),                   -exp(-4*t)+exp(-3*t), 1/2*exp(-
5*t)-exp(-4*t)+1/2*exp(-3*t)]
[                                0,                              exp(-4*t),
        -exp(-5*t)+exp(-4*t)]
[                                0,                                      0,
                   exp(-5*t)]

>> expm(t*Diag)

ans =

[ exp(-3*t),          0,          0]
[         0,  exp(-4*t),          0]
[         0,          0,  exp(-5*t)]
```

$$= e^{\begin{pmatrix} -3 & & 0 \\ & -4 & \\ 0 & & -5 \end{pmatrix} t}$$

```
>> T* expm(t*Diag)*inv(T)

ans =

[                        exp(-3*t),                   -exp(-4*t)+exp(-3*t), 1/2*exp(-
5*t)-exp(-4*t)+1/2*exp(-3*t)]
[                                0,                              exp(-4*t),
        -exp(-5*t)+exp(-4*t)]
[                                0,                                      0,
                   exp(-5*t)]
```

Example 7.10

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```
Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

To get started, select "MATLAB Help" from the Help menu.

>> A=[0 1 0;0 0 1;-27 -27 -9]

A =

     0     1     0
     0     0     1
   -27   -27    -9

>> [v j]=eig(A)

v =

  0.1048 + 0.0000i   0.1048 - 0.0000i   0.1048
 -0.3145 - 0.0000i  -0.3145 + 0.0000i  -0.3145
  0.9435             0.9435             0.9435
```

Three eigenvectors are l.d. V is a singular matrix. V^{-1} does not exist

```
j =

 -3.0000 + 0.0000i        0                  0
       0            -3.0000 - 0.0000i        0
       0                  0            -3.0000
```

Eigenvalues are repeated

```
>> [v j]=jordan(A)

v =

     9     3     1
   -27     0     0
    81   -27     0
```

Eigenvectors

Gen. Eigenvectors

V is a nonsingular invertible matrix.

```
j =

    -3     1     0
     0    -3     1
     0     0    -3
```

Jordan Canonical Form.

```
>> v1=v(:,1);
>> v2=v(:,2);
>> v3=v(:,3);
>> A*v1+3*v1

ans =

     0
     0
     0

>> A*v2+3*v2-v1

ans =

     0
```

$$Av_1 = -3v_1$$

$$Av_2 = -3v_2 + v_1$$

$$Av_3 = -3v_3 + v_2$$

$$V^{-1}AV = j$$

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```
     0
     0

>> A*v3+3*v3-v2

ans =

     0
     0
     0

>> syms t
>> H1=expm(t*j)

H1 =

[           exp(-3*t),        t*exp(-3*t), 1/2*t^2*exp(-3*t)]
[                  0,           exp(-3*t),       t*exp(-3*t)]
[                  0,                   0,         exp(-3*t)]

>> H2=v*H1*inv(v)

H2 =

[   9/2*t^2*exp(-3*t)+3*t*exp(-3*t)+exp(-3*t),                       t*exp(-3*t)+3*t^2*exp(-3*t), 
                          1/2*t^2*exp(-3*t)]
[                        -27/2*t^2*exp(-3*t),    exp(-3*t)+3*t*exp(-3*t)-9*t^2*exp(-3*t), 
              t*exp(-3*t)-3/2*t^2*exp(-3*t)]
[             81/2*t^2*exp(-3*t)-27*t*exp(-3*t),              -27*t*exp(-3*t)+27*t^2*exp(-3*t), 
 -6*t*exp(-3*t)+exp(-3*t)+9/2*t^2*exp(-3*t)]

>> H3=expm(t*A)

H3 =

[   9/2*t^2*exp(-3*t)+3*t*exp(-3*t)+exp(-3*t),                       t*exp(-3*t)+3*t^2*exp(-3*t), 
                          1/2*t^2*exp(-3*t)]
[                        -27/2*t^2*exp(-3*t),    exp(-3*t)+3*t*exp(-3*t)-9*t^2*exp(-3*t), 
              t*exp(-3*t)-3/2*t^2*exp(-3*t)]
[             81/2*t^2*exp(-3*t)-27*t*exp(-3*t),              -27*t*exp(-3*t)+27*t^2*exp(-3*t), 
 -6*t*exp(-3*t)+exp(-3*t)+9/2*t^2*exp(-3*t)]

>>
```

$$e^{\begin{pmatrix}-3 & 1 & 0\\ 0 & -3 & 1\\ 0 & 0 & -3\end{pmatrix}t}$$

$$e^{At} = V e^{jt} V^{-1} = V e^{\begin{pmatrix}-3 & 1 & 0\\ 0 & -3 & 1\\ 0 & 0 & -3\end{pmatrix}t} V^{-1}$$

Note the presence of $e^{-3t}, te^{-3t}, t^2e^{-3t}$

Example 7.8

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```
  Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> A=[0 2 1/3;0 -3 0;-27 -18 -6]

A =

         0    2.0000    0.3333
         0   -3.0000         0
  -27.0000  -18.0000   -6.0000

>> [v j]=eig(A)

v =

  -0.1104 - 0.0000i  -0.1104 + 0.0000i  -0.4395
        0                  0              0.5395
   0.9939             0.9939              0.7181
```

Two eigenvectors are repeated.

V is singular.

```
j =

  -3.0000 + 0.0000i        0                    0
        0            -3.0000 - 0.0000i          0
        0                  0              -3.0000

>> [v j]=jordan(A)

v =

    3.0000    0.3333   -0.6667
         0    1.0000    1.0000
  -27.0000         0         0
```

Two l.i. eigenvector

One generalized eigenvector.

```
j =

    -3     1     0
     0    -3     0
     0     0    -3
```

Jordan canonical form.

```
>> v1=v(:,1);
>> v2=v(:,2);
>> v3=v(:,3);
>> A*v1+3*v1

ans =

     0
     0
     0

>> A*v3+3*v3

ans =

     0
```

$$Av_1 = -3v_1$$

$$Av_2 = -3v_2 + v_1$$

$$Av_3 = -3v_3$$

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```
     0
     0

>> A*v2+3*v2-v1

ans =

     0
     0
     0

>> syms t
>> H1=expm(t*j)

H1 =

[   exp(-3*t), t*exp(-3*t),           0]
[           0,   exp(-3*t),           0]
[           0,           0,   exp(-3*t)]
```

$$e^{\begin{pmatrix}-3 & 1 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{pmatrix}t} = \left(\begin{array}{cc|c} e^{-3t} & te^{-3t} & 0\\ 0 & e^{-3t} & 0\\ \hline 0 & 0 & e^{-3t}\end{array}\right)$$

```
>> h2=v*H1*inv(v)

h2 =

[ 3*t*exp(-3*t)+exp(-3*t),              2*t*exp(-3*t),           1/3*t*exp(-3*t)]
[                       0,                  exp(-3*t),                         0]
[          -27*t*exp(-3*t),           -18*t*exp(-3*t), exp(-3*t)-3*t*exp(-3*t)]

>> H3=expm(t*A)

H3 =

[ 3*t*exp(-3*t)+exp(-3*t),              2*t*exp(-3*t),           1/3*t*exp(-3*t)]
[                       0,                  exp(-3*t),                         0]
[          -27*t*exp(-3*t),           -18*t*exp(-3*t), exp(-3*t)-3*t*exp(-3*t)]

>>
```

e^{At}

Note the presence of e^{-3t}, te^{-3t} and the absence of t^2e^{-3t} term.

Block Matrix:

Example 7·9:

$$A = \left(\begin{array}{cc|cc} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ \hline 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

$$e^{At} = \left(\begin{array}{cc|cc} e^{-3t} & te^{-3t} & \multicolumn{2}{c}{0} \\ 0 & e^{-3t} & & \\ \hline \multicolumn{2}{c|}{} & e^{-3t} & te^{-3t} \\ \multicolumn{2}{c|}{0} & 0 & e^{-3t} \end{array}\right)$$

Example 7·10:

$$A = \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{-3t} & te^{-3t} & \frac{t^2}{2}e^{-3t} & \frac{t^3}{6}e^{-3t} \\ 0 & e^{-3t} & te^{-3t} & \frac{t^2}{2}e^{-3t} \\ 0 & 0 & e^{-3t} & te^{-3t} \\ 0 & 0 & 0 & e^{-3t} \end{pmatrix}$$

Example 7.11

$$A_1 = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} A_1 & I \\ 0 & A_1 \end{pmatrix} \qquad Q: e^{At} = ??$$

Define:

$$B = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

Note that B & C commute i.e. $BC = CB$.

$$= \begin{pmatrix} 0 & A_1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore e^{At} = e^{(B+C)t} = e^{Bt} e^{Ct}$$

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Since C is nilpotent i.e. $C^2=0$

$$e^{Ct} = I + Ct$$

$$= \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & tI \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} I & tI \\ 0 & I \end{pmatrix}$$ ← Each entry is a 2x2 matrix.

$$e^{Bt} = \begin{pmatrix} e^{A_1 t} & 0 \\ 0 & e^{A_1 t} \end{pmatrix}$$

$\therefore$ $e^{At} =$

$$\begin{pmatrix} e^{A_1 t} & 0 \\ 0 & e^{A_1 t} \end{pmatrix} \begin{pmatrix} I & tI \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} e^{A_1 t} & te^{A_1 t} \\ 0 & e^{A_1 t} \end{pmatrix}$$

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$$e^{A_1 t} = \begin{pmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{pmatrix}$$

$$\therefore \; e^{At} =$$

$$\begin{pmatrix} e^{-3t} & te^{-3t} & te^{-3t} & t^2e^{-3t} \\ 0 & e^{-3t} & 0 & te^{-3t} \\ 0 & 0 & e^{-3t} & te^{-3t} \\ 0 & 0 & 0 & e^{-3t} \end{pmatrix}$$

```
Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

To get started, select "MATLAB Help" from the Help menu.

>> A1=[-3 1;0 -3]

A1 =

  -3     1
   0    -3

>> I=[1 0;0 1]

I =

   1     0
   0     1

>> A=[A1 I;0 A1]
??? Error using ==> horzcat
All matrices on a row in the bracketed expression must have the
 same number of rows.

>> A=[A1 I;0*I A1]

A =

  -3     1     1     0
   0    -3     0     1
   0     0    -3     1
   0     0     0    -3

>> syms t
>> H1=expm(t*A)

H1 =

[       exp(-3*t),     t*exp(-3*t),     t*exp(-3*t), t^2*exp(-3*t)]
[               0,       exp(-3*t),               0,   t*exp(-3*t)]
[               0,               0,       exp(-3*t),   t*exp(-3*t)]
[               0,               0,               0,     exp(-3*t)]

>> [v j]=jordan(A)

v =

   2.0000        0         0         0
        0   1.0000    0.5000    0.5000
        0   1.0000   -0.5000   -0.5000
        0        0    1.0000        0

j =

  -3     1     0     0
   0    -3     1     0
   0     0    -3     0
   0     0     0    -3
```

7.22

This is not in a jordan form.

e^{At}

2 l.i. eigenvectors.

generalized eigenvectors

Jordan form

7.23

```
>> v1=v(:,1);
>> v2=v(:,2);
>> v3=v(:,3);
>> v4=v(:,4);
>> A*v1+3*v1

ans =

     0
     0
     0
     0

>> A*v4+3*v4

ans =

     0
     0
     0
     0

>> A*v2+3*v2-v1

ans =

     0
     0
     0
     0

>> A*v3+3*v3-v2

ans =

     0
     0
     0
     0

>> H2=expm(t*j)

H2 =

[          exp(-3*t),         t*exp(-3*t), 1/2*t^2*exp(-3*t),                   0]
[                  0,           exp(-3*t),        t*exp(-3*t),                   0]
[                  0,                   0,          exp(-3*t),                   0]
[                  0,                   0,                  0,           exp(-3*t)]

>> v*H2*inv(v)
```

$= e^{At}$

```
ans =

[      exp(-3*t),    t*exp(-3*t),    t*exp(-3*t), t^2*exp(-3*t)]
[              0,      exp(-3*t),              0,   t*exp(-3*t)]
[              0,              0,      exp(-3*t),   t*exp(-3*t)]
[              0,              0,              0,     exp(-3*t)]
```

Example 7.12

7.24

```
  Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> syms t
>> syms sigma
>> syms omega
>> A1=[sigma omega;-omega sigma]

A1 =

  sigma,   omega
 -omega,   sigma

>> I=[1 0;0 1]

I =

     1     0
     0     1

>> A=[A1 I;0*I A1]

A =

[  sigma,   omega,       1,       0]
[ -omega,   sigma,       0,       1]
[      0,       0,   sigma,   omega]
[      0,       0,  -omega,   sigma]

>> H1=expm(t*A)

H1 =

[    exp(t*sigma)*cos(t*omega),     exp(t*sigma)*sin(t*omega),  t*exp(t*sigma)*cos(t*omega
),  t*exp(t*sigma)*sin(t*omega)]
[   -exp(t*sigma)*sin(t*omega),     exp(t*sigma)*cos(t*omega), -t*exp(t*sigma)*sin(t*omega
),  t*exp(t*sigma)*cos(t*omega)]
[                             0,                             0,    exp(t*sigma)*cos(t*omega
),    exp(t*sigma)*sin(t*omega)]
[                             0,                             0,   -exp(t*sigma)*sin(t*omega
),    exp(t*sigma)*cos(t*omega)]

>> B=A;
>> B(1,3)=0;
>> B(2,4)=0;
>> B(2,3)=1;
>> B

B =

[  sigma,   omega,       0,       0]
[ -omega,   sigma,       1,       0]
[      0,       0,   sigma,   omega]
[      0,       0,  -omega,   sigma]

>> H2=expm(t*B)
```

$$A1 = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

$$A = \left(\begin{array}{cc|cc} \sigma & \omega & 1 & 0 \\ -\omega & \sigma & 0 & 1 \\ \hline 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{array}\right)$$

Jordan block for repeated complex eigenvalues.

$$e^{At} = \left(\begin{array}{cc|cc} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t & te^{\sigma t}\cos\omega t & te^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t & -te^{\sigma t}\sin\omega t & te^{\sigma t}\cos\omega t \\ \hline 0 & 0 & e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ 0 & 0 & -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{array}\right)$$

This is not in the Jordan form.

H2 =

7.25

$$e^{Bt} = \left(\begin{array}{cc|cc} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t & \frac{t}{2}e^{\sigma t}\sin\omega t & -\frac{\omega t}{2}e^{\sigma t}\cos\omega t + \frac{1}{2\omega}e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t & \frac{\omega t}{2}e^{\sigma t}\cos\omega t + \frac{1}{2\omega}e^{\sigma t}\sin\omega t & \frac{t}{2}e^{\sigma t}\sin\omega t \\ \hline 0 & 0 & e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ 0 & 0 & -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{array}\right)$$

```
>> [v j]=jordan(B)

v =

[          -1/4, -1/4*i/omega,          -1/4,  1/4*i/omega]
[        -1/4*i,            0,         1/4*i,            0]
[             0,       -1/2*i,             0,        1/2*i]
[             0,          1/2,             0,          1/2]


j =

[ sigma+i*omega,             1,             0,             0]
[             0, sigma+i*omega,             0,             0]
[             0,             0, sigma-i*omega,             1]
[             0,             0,             0, sigma-i*omega]
```

Repeated eigenvalues at $\sigma \pm i\omega$

$\boxed{\text{complex jordan blocks}}$

Remark:

The matrices

$$A_1 = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} \ \& \ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = Z$$

do not commute. Note that

$$A_1 Z = \begin{pmatrix} \omega & 0 \\ \sigma & 0 \end{pmatrix}, \quad Z A_1 = \begin{pmatrix} 0 & 0 \\ \sigma & \omega \end{pmatrix}$$

Thus you cannot do the following

$$\underbrace{\begin{pmatrix} \sigma & \omega & 0 & 0 \\ -\omega & \sigma & 1 & 0 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} \sigma & \omega & 0 & 0 \\ -\omega & \sigma & 0 & 0 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{pmatrix}}_{L_1} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{L_2}$$

$$\Rightarrow e^{Bt} = e^{(L_1 + L_2)t} \overset{?}{=} e^{L_1 t} e^{L_2 t}$$

Not true.

In order to gain insight to the problem of computing e^{Bt} (page 7.25), we would like to reduce the matrix

$$\begin{pmatrix} \sigma & \omega & 0 & 0 \\ -\omega & \sigma & 1 & 0 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{pmatrix}$$

to the Jordan canonical form

$$\begin{pmatrix} \sigma & \omega & 1 & 0 \\ -\omega & \sigma & 0 & 1 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{pmatrix}$$

by a similarity transformation.

Need to solve

$$\begin{pmatrix} \sigma & \omega & 0 & 0 \\ -\omega & \sigma & 1 & 0 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} = $$

A_1, Z

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} \begin{pmatrix} \sigma & \omega & 1 & 0 \\ -\omega & \sigma & 0 & 1 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{pmatrix}$$

The equations we need to solve are

$$A_1 T_1 + Z T_3 = T_1 A_1 \qquad (1)$$

$$A_1 T_2 + Z T_4 = T_1 + T_2 A_1 \qquad (2)$$

$$A_1 T_3 = T_3 A_1 \qquad (3)$$

$$A_1 T_4 = T_3 + T_4 A_1 \qquad (4)$$

7.29

Writing

$$T_3 = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}, \quad T_1 = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

$$T_4 = \begin{pmatrix} l_{11} & l_{12} \\ +l_{21} & l_{22} \end{pmatrix}, \quad T_2 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

From 3 it follows that

$$\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

$$\Updownarrow$$

$$\sigma t_{11} + \omega t_{21} = t_{11}\sigma + t_{12}(-\omega) \Leftrightarrow t_{12} = -t_{21}$$

$$\sigma t_{12} + \omega t_{22} = t_{11}\omega + t_{12}\sigma \Leftrightarrow t_{11} = t_{22}$$

$$\left.\begin{aligned} -\omega t_{11} + \sigma t_{21} &= t_{21}\sigma + t_{22}(-\omega) \\ -\omega t_{12} + \sigma t_{22} &= t_{21}\omega + t_{22}\sigma \end{aligned}\right\} \Leftarrow$$

$$T_3 = \begin{pmatrix} t_{11} & t_{12} \\ -t_{12} & t_{11} \end{pmatrix}$$

From ① it follows that

$$\underbrace{\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}}_{A_1 T_1} = \underbrace{\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}}_{T_1 A_1} - \underbrace{\begin{pmatrix} 0 & 0 \\ t_{11} & t_{12} \end{pmatrix}}_{z T_3}$$

$$\Updownarrow$$

$$\sigma s_{11} + \omega s_{21} = s_{11}\sigma + s_{12}(-\omega) \Leftrightarrow s_{21} = -s_{12}$$

$$\sigma s_{12} + \omega s_{22} = s_{11}\omega + s_{12}\sigma \Leftrightarrow s_{11} = s_{22}$$

$$-\omega s_{11} + \sigma s_{21} = s_{21}\sigma - \omega s_{22} - t_{11} \Leftrightarrow t_{11} = 0$$

$$-\omega s_{12} + \sigma s_{22} = s_{21}\omega + s_{22}\sigma - t_{12} \Leftrightarrow t_{12} = 0$$

Thus

$$T_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} s_{11} & s_{12} \\ -s_{12} & s_{11} \end{pmatrix}$$

From 4 we write

$$A_1 T_4 = T_4 A_1$$

It follows from an argument similar to page 7.30 that

$$T_4 = \begin{pmatrix} l_{11} & l_{12} \\ -l_{12} & l_{11} \end{pmatrix}$$

Finally from 2 we have

$$\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ l_{11} & l_{12} \end{pmatrix} =$$

$$\begin{pmatrix} s_{11} & s_{12} \\ -s_{12} & s_{11} \end{pmatrix} + \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

7.32

$$\sigma m_{11} + \omega m_{21} = s_{11} + m_{11}\sigma - \omega m_{12}$$

$$\Downarrow$$

$$s_{11} = \omega(m_{12} + m_{21})$$

$$\sigma m_{12} + \omega m_{22} = s_{12} + m_{11}\omega + m_{12}\sigma$$

$$\Downarrow$$

$$s_{12} = \omega(m_{22} - m_{11})$$

$$\begin{aligned} -\omega m_{11} + \sigma m_{21} + l_{11} &= \underbrace{-s_{12}}_{\parallel} + m_{21}\sigma - \omega m_{22} \\ &= -\omega m_{22} + \omega m_{11} \\ &\quad + m_{21}\sigma - \omega m_{22} \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} l_{11} &= 2\omega m_{11} - 2\omega m_{22} \\ &= 2\omega(m_{11} - m_{22}) \end{aligned}$$

$$\begin{aligned} -\omega m_{12} + \sigma m_{22} + l_{12} &= \underbrace{s_{11}}_{\parallel} + m_{21}\omega + m_{22}\sigma \\ &= \omega m_{12} + \omega m_{21} + m_{21}\omega + m_{22}\sigma \end{aligned}$$

$$\Downarrow$$

$$l_{12} = 2\omega m_{21} + 2\omega m_{12} = 2\omega(m_{12} + m_{21})$$

Writing $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ as follows

$$\left(\begin{array}{cc|cc} \omega(m_{12}+m_{21}) & \omega(m_{22}-m_{11}) & m_{11} & m_{12} \\ -\omega(m_{22}-m_{11}) & \omega(m_{12}+m_{21}) & m_{21} & m_{22} \\ \hline 0 & 0 & 2\omega(m_{11}-m_{22}) & 2\omega(m_{12}+m_{21}) \\ 0 & 0 & -2\omega(m_{12}+m_{21}) & 2\omega(m_{11}-m_{22}) \end{array}\right) = T$$

$\boxed{\text{End of Example 7.12}}$

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```
  Using Toolbox Path Cache.  Type "help toolbox_path_cache" for more info.

  To get started, select "MATLAB Help" from the Help menu.

>> syms sigma omega
>> syms t a b c d
>> A1=[sigma omega;-omega sigma]

A1 =

[  sigma,  omega]
[ -omega,  sigma]

>> I=[1 0;0 1]

I =
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
>> A=[A1 I;0*I A1]

A =

[  sigma,  omega,      1,      0]
[ -omega,  sigma,      0,      1]
[      0,      0,  sigma,  omega]
[      0,      0, -omega,  sigma]

>> Z=[0 0;1 0]

Z =
```

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

```
>> B=[A1 Z;0*I A1]

B =

[  sigma,  omega,      0,      0]
[ -omega,  sigma,      1,      0]
[      0,      0,  sigma,  omega]
[      0,      0, -omega,  sigma]
```

(7·35)

```
T =

[        omega*b+omega*c,        omega*d-omega*a,                        a,                       b]
[       -omega*d+omega*a,        omega*b+omega*c,                        c,                       d]
[                      0,                      0,   2*omega*a-2*omega*d,   2*omega*b+2*omega*c]
[                      0,                      0,  -2*omega*b-2*omega*c,   2*omega*a-2*omega*d]

>> H1=T*A

H1 =

[           (omega*b+omega*c)*sigma-(omega*d-omega*a)*omega,                   (omega*b+omega*c)*om
ega+(omega*d-omega*a)*sigma,                                                   omega*c+a*sigma,
                                                   omega*d+b*sigma]
[          (-omega*d+omega*a)*sigma-(omega*b+omega*c)*omega,                  (-omega*d+omega*a)*om
ega+(omega*b+omega*c)*sigma,                                       -2*omega*d+omega*a+c*sigma,
                                       omega*b+2*omega*c+d*sigma]
[                                                                        0,
                                      0,  (2*omega*a-2*omega*d)*sigma-(2*omega*b+2*omega*c)*omega,  (
2*omega*a-2*omega*d)*omega+(2*omega*b+2*omega*c)*sigma]
[                                                                        0,
                                      0,  (-2*omega*b-2*omega*c)*sigma-(2*omega*a-2*omega*d)*omega,  (-
2*omega*b-2*omega*c)*omega+(2*omega*a-2*omega*d)*sigma]

>> H2=B*T

H2 =

[          (-omega*d+omega*a)*omega+(omega*b+omega*c)*sigma,                   (omega*b+omega*c)*om
ega+(omega*d-omega*a)*sigma,                                                   omega*c+a*sigma,
                                                   omega*d+b*sigma]
[          (-omega*d+omega*a)*sigma-(omega*b+omega*c)*omega,                   (omega*b+omega*c)*si
gma-(omega*d-omega*a)*omega,                                       -2*omega*d+omega*a+c*sigma,
                                       omega*b+2*omega*c+d*sigma]
[                                                                        0,
                                      0,  (-2*omega*b-2*omega*c)*omega+(2*omega*a-2*omega*d)*sigma,  (
2*omega*a-2*omega*d)*omega+(2*omega*b+2*omega*c)*sigma]
[                                                                        0,
                                      0,  (-2*omega*b-2*omega*c)*sigma-(2*omega*a-2*omega*d)*omega,  (
2*omega*a-2*omega*d)*sigma-(2*omega*b+2*omega*c)*omega]

>> H1-H2

ans =

[          -(omega*d-omega*a)*omega-(-omega*d+omega*a)*omega,
                                      0,                                                          0,
                                                                        0]
[                                                                        0,           (-omega*d+omega*a)*
omega+(omega*d-omega*a)*omega,                                                                    0,
                                                                        0]
[                                                                        0,
                                      0,  -(2*omega*b+2*omega*c)*omega-(-2*omega*b-2*omega*c)*omega,
                                                                        0]
[                                                                        0,
                                      0,                                                          0,
```

```
(-2*omega*b-2*omega*c)*omega+(2*omega*b+2*omega*c)*omega]

>>
```

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H1 - H2 is the zero matrix.

Hence

$$A = T^{-1} B T$$

or

A & B are similar

7.37 ~~7.34~~

Additional Example:

This matrix is in the Jordan form.

```
C =

[  sigma,  omega,      0,      0]
[ -omega,  sigma,      0,      0]
[      0,      0,  sigma,  omega]
[      0,      0, -omega,  sigma]

>> [v j]=jordan(C)

v =

[    1/2,    1/2,      0,      0]
[  1/2*i, -1/2*i,      0,      0]
[    1/2,    1/2,    1/2,    1/2]
[  1/2*i, -1/2*i,  1/2*i, -1/2*i]


j =

[ sigma+i*omega,             0,             0,             0]
[             0, sigma-i*omega,             0,             0]
[             0,             0, sigma+i*omega,             0]
[             0,             0,             0, sigma-i*omega]
```

The matrices

$$\begin{pmatrix} \sigma & \omega & 0 & 0 \\ -\omega & \sigma & 0 & 0 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{pmatrix} \ \& \ \begin{pmatrix} \sigma & \omega & 1 & 0 \\ -\omega & \sigma & 0 & 1 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{pmatrix}$$

are <u>not</u> similar.

Additional Example:

Not in the Jordan form.

```
D =

[  sigma,  omega,      0,      1]
[ -omega,  sigma,      0,      0]
[      0,      0,  sigma,  omega]
[      0,      0, -omega,  sigma]

>> [v j]=jordan(D)

v =

[          1/4, -1/4*i/omega,          1/4,  1/4*i/omega]
[        1/4*i,            0,       -1/4*i,            0]
[            0,       -1/2*i,            0,        1/2*i]
[            0,          1/2,            0,          1/2]


j =

[ sigma+i*omega,             1,             0,             0]
[             0, sigma+i*omega,             0,             0]
[             0,             0, sigma-i*omega,             1]
[             0,             0,             0, sigma-i*omega]

>> expm(t*j)

ans =

[      exp(t*sigma)*cos(t*omega)+i*exp(t*sigma)*sin(t*omega), t*exp(t*sigma)*cos(t*omega)+
i*t*exp(t*sigma)*sin(t*omega),                                                              0,
                                                            0]
[                                                             0,      exp(t*sigma)*cos(t*omega
)+i*exp(t*sigma)*sin(t*omega),                                                              0,
                                                            0]
[                                                             0,
                                 0,      exp(t*sigma)*cos(t*omega)-i*exp(t*sigma)*sin(t*omega),
 t*exp(t*sigma)*cos(t*omega)-i*t*exp(t*sigma)*sin(t*omega)]
[                                                             0,
                                 0,                                                         0,
     exp(t*sigma)*cos(t*omega)-i*exp(t*sigma)*sin(t*omega)]
```

Jordan form is

$$\begin{pmatrix} \sigma & \omega & 1 & 0 \\ -\omega & \sigma & 0 & 1 \\ 0 & 0 & \sigma & \omega \\ 0 & 0 & -\omega & \sigma \end{pmatrix}$$

Matrix with complex entries